Generalized Formulation of Nonlinear Pitch-Yaw-Roll Coupling: Part I—Nonaxisymmetric Bodies

MURRAY TOBAK* AND LEWIS B. SCHIFF†
NASA Ames Research Center, Moffett Field, Calif.

A nonlinear aerodynamic moment system, formulated in a previous study to apply to arbitrary motions of bodies of revolution in free flight, is generalized to include nonaxisymmetric bodies (e.g., aircraft) within a uniform treatment. Within the assumption of a linear dependence on coning rate, the total moment is shown to be compounded of the contributions from four simple motions and a clear physical meaning is attached to each contribution. Applied to aircraft motions, the formulation includes the nonlinear interactions between motions that normally are excluded in the classical treatment. The results may be useful in the design of experiments and the interpretation of results obtained therefrom.

Nomenclature

C_A = axial-force coefficient (along x_B), 2 (axial force)/$\rho V^2 S$
C_Y = side-force coefficient in the aerodynamic axis system, (along y), 2 (side force)/$\rho V^2 S$
C_Z = normal-force coefficient in the aerodynamic axis system (along z), 2 (normal force)/$\rho V^2 S$
$\hat{C}_Y, \hat{C}_Z$ = side-force and normal-force coefficients in the body axis system; along y_B, z_B, respectively
C_l = rolling-moment coefficient in the aerodynamic axis system (along x_B), $2\bar{L}/\rho V^2 Sl$
C_m = pitching-moment coefficient in the aerodynamic axis system (along y), $2\bar{M}/\rho V^2 Sl$
C_n = side-moment coefficient in the aerodynamic axis system (along z), $2\bar{N}/\rho V^2 Sl$
$\hat{C}_l, \hat{C}_m, \hat{C}_n$ = rolling, pitching, and yawing-moment coefficients in the body axis system; along x_B, y_B, z_B, respectively
$G[\delta(\xi), \psi(\xi), \lambda(\xi), q(\xi), r(\xi)]$ = functional notation: value at $\xi = t$ of a time-dependent function which depends on all values taken by the five argument functions $\delta(\xi)$, $\psi(\xi)$, $\lambda(\xi)$, $q(\xi)$, $r(\xi)$ over the time interval $0 \leq \xi \leq t$
$\bar{L}$ = moment along longitudinal axis of body, Fig. 1
l = reference length
$\bar{M}$ = moment along an axis normal to the plane of the resultant angle of attack (along y), Fig. 1
$\bar{N}$ = moment along an axis in the plane of the resultant angle of attack (along z), Fig. 1
p_B, q_B, r_B = components along the x_B, y_B, z_B axes, respectively, of the total angular velocity of the body axes relative to inertial space
q, r = components of angular velocity along the y, z axes, respectively, Eq. (5)
$\tilde{q}, \tilde{r}$ = components of angular velocity along the $\tilde{y}, \tilde{z}$ axes, respectively, Eq. (4)
S = reference area
t = time
u_B, v_B, w_B = components of flight velocity along x_B, y_B, z_B axes, respectively, Fig. 1
V = magnitude of flight velocity vector
x_B, y_B, z_B = body-fixed axes, origin at center of gravity, x_B coincident with a longitudinal axis of the body, Fig. 1
x_B, y, z = aerodynamic axes, origin at center of gravity, x_B, z in the plane of the resultant angle of attack, y, z in the crossflow plane normal to the resultant angle-of-attack plane, Fig. 1
$x_B, \tilde{y}, \tilde{z}$ = nonrolling axes (with respect to inertial space), origin at center of gravity, $\tilde{y}$, $\tilde{z}$ in the crossflow plane, Fig. 1
α = angle of attack in body axes, Eq. (7)
$\hat{\alpha}$ = angle-of-attack parameter in body axis system, w_B/V
β = angle of sideslip in body axes, Eq. (7)
$\hat{\beta}$ = angle-of-sideslip parameter in body axis system, v_B/V
γ = dimensionless axial component of velocity, Fig. 1 and Eq. (2)
δ = magnitude of the dimensionless crossflow velocity in the aerodynamic axis system, Fig. 1 and Eq. (2)
ε = $\tan\sigma$, Fig. 1 and Eq. (2)
λ = angular inclination from the $\tilde{y}$ axis of the crossflow velocity vector, Fig. 1
ρ = atmospheric mass density
σ = resultant angle of attack defined by x_B axis and velocity vector, Fig. 1
$\dot{\phi}$ = coning rate of x_B axis about the velocity vector of a body in level flight
$\tilde{\phi}$ = angular inclination from the $\tilde{y}$ axis of the y_B axis, Fig. 1 and Eq. (1)
ψ = angular inclination from the crossflow velocity vector of the z_B axis, Fig. 1
$\omega_1, \omega_2, \omega_3$ = components of vector $[\dot{\lambda}, q, r]$ resolved so that ω_1 is directed along the flight velocity vector, Eq. (2), Part II
$\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3$ = components of vector $[p_B, q_B, r_B]$ resolved so that $\hat{\omega}_1$ is directed along the flight velocity vector, Eq. (10), Part II
$(\dot{\ })$ = $d/dt(\ )$

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* Research Scientist. Member AIAA.

† Research Scientist. Associate Fellow AIAA.

Introduction

IN a series of papers beginning with Ref. 1, concepts from nonlinear functional analysis were used as the basis for formulating aerodynamic moment systems that do not depend on a linearity assumption (and thereby, are not implicitly based on a principle of superposition). The latest of this series[2] proposed a moment system applicable to arbitrary motions of bodies of revolution in free flight. The total moment was shown to be compounded of the contributions from four simple motions and a clear physical meaning was attached to each contribution. Further, the moment system was shown to be compatible with one proposed earlier by Murphy,[3,4] both illustrating the importance of coupling terms that are missed by nonlinear extrapolations of the linear formulation.

The purpose of Part I of this paper is to generalize the previous analysis for bodies of revolution to include nonaxisymmetric bodies (e.g., aircraft) within a uniform treatment. By only a slight change in the functional dependence, it will be shown that the previous analysis can be made applicable, and that the total moment still may be compounded of contributions from four simple motions.

Analysis

Nonlinear Formulation of Aerodynamic Moment System

Much of the analysis can be carried over from the previous study of bodies of revolution[2] by studying the flow in a crossflow plane referred to a nonrolling axis system.

Coordinate systems

Three coordinate systems will be used. They have a common origin at the center of gravity and a common axis x_B alined with a longitudinal axis of the body. The longitudinal axis no longer need be an axis of cylindrical symmetry. Axes x_B, y_B, z_B are body-fixed axes. The plane formed by y_B, z_B is the crossflow plane, illustrated in Fig. 1(a). Axes x_B, $\tilde{y}$, $\tilde{z}$ are nonrolling with respect to inertial space. The angle $\tilde{\phi}$ through which the body axes have rolled at any time t can be defined relative to the nonrolling axis system as

$$\tilde{\phi} = \int_0^t p_B \, d\tau \tag{1}$$

where p_B is the body roll rate; that is, the component of angular velocity along the x_B axis. The vector with magnitude δ is the projection of the resultant angle of attack in the crossflow plane; it will be called the dimensionless crossflow velocity vector. Reference to Fig. 1(b) gives

$$\begin{aligned} \delta &= [(v_B/V)^2 + (w_B/V)^2]^{1/2} = \sin\sigma \\ \gamma &= u_B/V = \cos\sigma \\ \varepsilon &= \delta/\gamma = \tan\sigma \end{aligned} \tag{2}$$

As before, λ is the angular inclination of the crossflow velocity vector measured relative to the nonrolling axis system; ψ is the angular inclination of the body axes from the crossflow velocity vector. The roll rate p_B is the sum

$$p_B = \dot{\lambda} + \dot{\psi} \tag{3}$$

while the remaining components of the angular velocity vector in the body axis system q_B, r_B are measured in the direction y_B, z_B, respectively. In the nonrolling axis system x_B, $\tilde{y}$, $\tilde{z}$ the angular velocity components are p_B, $\tilde{q}$, $\tilde{r}$, respectively, where $\tilde{q}$ and $\tilde{r}$ are related to q_B and r_B by

$$\tilde{q} + i\tilde{r} = e^{i\tilde{\phi}}(q_B + ir_B) \tag{4}$$

Finally, an axis system x_B, y, z will be called the aerodynamic axis system. Axis z lies in the crossflow plane and is alined with the direction of δ; axis y lies in the crossflow plane alined with an axis normal to the direction of δ. The angular velocity components q, r in the aerodynamic axis system are related to q_B, r_B in the body axis system through

$$q + ir = e^{i\psi}(q_B + ir_B) \tag{5}$$

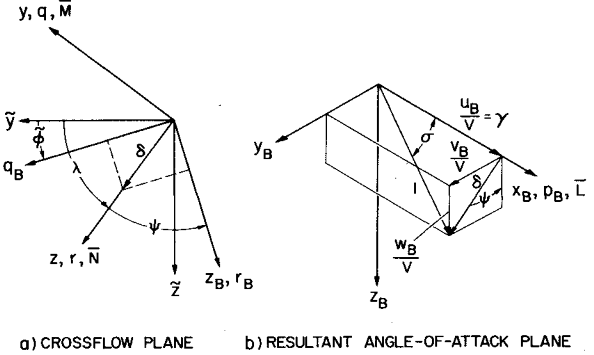

Fig. 1 Axes, angles, and velocity components in the crossflow and resultant angle-of-attack planes.

Functional dependence

Forces and moments will be defined first relative to the aerodynamic axis system. The moment along the y axis (i.e., along an axis normal to the plane of the resultant angle of attack) will be called the pitching moment $\bar{M}$. The moment along the z axis (i.e., along an axis in the plane of the resultant angle of attack) will be called the side moment $\bar{N}$. The moment along x_B will be called the rolling moment $\bar{L}$.

The principal change in the analysis occasioned by the elimination of axial symmetry occurs at this point. For the body of revolution, it was assumed that the moment coefficients C_l, C_m, C_n would be functionals of the variables δ, q, r, $\dot{\lambda}$, and $\dot{\psi}$. The choice of $\dot{\lambda}$ and $\dot{\psi}$ as argument functions rather than λ and ψ was dictated by the recognition that, with axial symmetry, the moments should be insensitive to the particular orientation ψ of the body relative to the crossflow velocity vector, as well as the particular orientation λ of the crossflow velocity vector relative to inertial space. While the latter condition will remain true even in the absence of axial symmetry, it is clear that the moments will now depend on the orientation of the body relative to the crossflow velocity vector. Thus, the absence of axial symmetry is accounted for by assuming that the moment coefficients will be functionals of the variables δ, q, r, $\dot{\lambda}$, and ψ, rather than $\dot{\psi}$. For example, the pitching-moment coefficient C_m is specified as a functional of the form

$$C_m(t) = G[\delta(\xi), \psi(\xi), \dot{\lambda}(\xi), q(\xi), r(\xi)] \tag{6}$$

It is noted [cf. Fig. 1(b)] that δ and ψ are no more than the polar coordinates of the dimensionless velocities w_B/V and v_B/V in the body axis system. Let w_B/V be called the angle-of-attack parameter $\hat{\alpha}$ and v_B/V the angle-of-sideslip parameter $\hat{\beta}$; $\hat{\alpha}$ and $\hat{\beta}$ are related to the standard NASA definitions of angle of attack α and angle of sideslip β through

$$\left.\begin{aligned} \tan\alpha &= w_B/u_B = \hat{\alpha}/\gamma \\ \sin\beta &= v_B/V = \hat{\beta} \end{aligned}\right\} \tag{7}$$

and to δ and ψ through

$$\hat{\alpha} + i\hat{\beta} = \delta\, e^{i\psi} \tag{8}$$

Hence, the dependence on δ and ψ in Eq. (6) is equivalent to dependence on $\hat{\alpha}$ and $\hat{\beta}$. To show the equivalence explicitly, an alternative form for the moment system will be developed later completely in terms of variables in the body axis system. Let $\hat{C}_l$, $\hat{C}_m$, $\hat{C}_n$ be moment coefficient components along x_B, y_B, z_B, respectively, and specify that, for example, $\hat{C}_m$ be a functional of the form

$$\hat{C}_m(t) = H[\hat{\alpha}(\xi), \hat{\beta}(\xi), p_B(\xi), q_B(\xi), r_B(\xi)] \tag{9}$$

The moment systems based on Eqs. (6) and (9) must of course be compatible and transferable one to the other through the relations

$$\left.\begin{aligned} C_l &= \hat{C}_l \\ C_m + iC_n &= e^{i\psi}(\hat{C}_m + i\hat{C}_n) \end{aligned}\right\} \tag{10}$$

Approximate formulation in the aerodynamic axis system

With the functional dependence specified by Eq. (6), the formulation of indicial responses and an integral form for $C_m(t)$ parallels that of Ref. 1. In contrast to the case of the body of revolution, where, in expanding the integral form, it was assumed that only δ would be large, here it must be assumed that both δ and ψ (or equivalently $\hat{\alpha}$ and $\hat{\beta}$) may be large. The rates $\dot{\lambda}$, q, and r may be assumed to remain small, however, and the integral form can be expanded about $\dot{\lambda} = 0$, $q = 0$, $r = 0$ to yield, to first order in the rates, a sum of stability derivatives. The result for $C_m(t)$ is

$$\begin{aligned} C_m(t) = {} & C_m(\infty; \delta(t), \psi(t)) + (\dot{\lambda}l/V)C_{m_{\dot{\lambda}}}(\infty; \delta(t), \psi(t)) + \\ & (ql/V)C_{m_q}(\infty; \delta(t), \psi(t)) + (rl/V)C_{m_r}(\infty; \delta(t), \psi(t)) + \\ & (\dot{\delta}l/V)C_{m_{\dot{\delta}}}(\delta(t), \psi(t)) + (\dot{\psi}l/V)C_{m_{\dot{\psi}}}(\delta(t), \psi(t)) \end{aligned} \tag{11}$$

where, in the functional dependence notation, the infinity symbol indicates steady flow and, for brevity, the zeros belonging to $\dot{\lambda}$, q, r have been omitted. Analogous expressions for C_n and C_l

and the axial, normal, and side-force coefficients C_A, C_Z, C_Y are obtained by substituting these coefficients wherever C_m appears in Eq. (11).

Simplification of the formulation in the aerodynamic axis system

As in Ref. 2, it is possible to simplify Eq. (11) by assuming that, with small plunging, terms multiplied by $(q-\dot{\sigma})$ and $(r-\varepsilon\dot{\lambda})$ will be negligibly small compared with the principal contributions. Adding and subtracting the terms $(\dot{\sigma}l/V)C_{m_q}$ and $\varepsilon(\dot{\lambda}l/V)C_{m_r}$ and rewriting Eq. (11) yields

$$\begin{aligned}C_m(t) = {} & C_m(\infty\,; \delta(t), \psi(t)) + (\dot{\psi}l/V)C_{m_{\dot{\psi}}}(\delta(t), \psi(t)) + \\ & (\dot{\sigma}l/V)\{C_{m_q}(\infty\,; \delta(t), \psi(t)) + \gamma C_{m_{\dot{\delta}}}(\delta(t), \psi(t))\} + \\ & (1/\gamma)(\dot{\lambda}l/V)\{\gamma C_{m_{\dot{\lambda}}}(\infty\,; \delta(t), \psi(t)) + \delta C_{m_r}(\infty\,; \delta(t), \psi(t))\} + \\ & (q-\dot{\sigma})(l/V)C_{m_q}(\infty\,; \delta(t), \psi(t)) + \\ & (r-\varepsilon\dot{\lambda})(l/V)C_{m_r}(\infty\,; \delta(t), \psi(t))\end{aligned} \tag{12}$$

The terms multiplied by $(q-\dot{\sigma})$ and $(r-\varepsilon\dot{\lambda})$ are discarded on the assumption of small plunging, since for zero plunging they vanish identically. The remaining terms are identified by comparing them with those obtained in the case of zero plunging where, exactly, $q = \dot{\sigma}$, $r = \delta\dot{\phi}$, $\dot{\lambda} = \gamma\dot{\phi}$, and $\dot{\phi}$ is the coning rate of the longitudinal axis around the flight velocity vector. The result is

$$\left.\begin{aligned}C_{m_{\dot{\sigma}}} &= C_{m_q} + \gamma C_{m_{\dot{\delta}}} \\ C_{m_{\dot{\phi}}} &= \gamma C_{m_{\dot{\lambda}}} + \delta C_{m_r}\end{aligned}\right\} \tag{13}$$

As before the term $C_{m_{\dot{\sigma}}}$ is the damping-in-pitch coefficient for planar pitching oscillations about an axis normal to the plane of σ, measured now, however, with *both* δ and ψ at fixed inclinations $\delta = \text{const}$, $\psi = \text{const}$. The term $C_{m_{\dot{\phi}}}$ is the rate of change with $\dot{\phi}l/V$, evaluated at $\dot{\phi} = 0$, of the pitching-moment coefficient that would be measured in a steady coning motion $\delta = \text{const}$, $\psi = \text{const}$, $\dot{\phi} = \text{const}$. The term $C_m(\infty\,; \delta(t), \psi(t))$ is the pitching-moment coefficient that would be measured in steady planar motion with δ and ψ at the fixed inclinations $\delta = \text{const}$, $\psi = \text{const}$. Thus, these three coefficients can be obtained from the same defining experiments as those for the body of revolution except that each of them must now be measured, in addition, over a range of inclinations $\psi = \text{const}$. The defining experiment for the remaining term, $C_{m_{\dot{\psi}}}$, requires revision, however. Since both δ and ψ must remain fixed, $C_{m_{\dot{\psi}}}$ no longer can be obtained from the classical Magnus experiment $\delta = \text{const}$, $\dot{\psi} = \text{const}$, $\dot{\phi} = 0$. Instead, $C_{m_{\dot{\psi}}}$ must now be measured from small oscillations in ψ about $\psi = \text{const}$ with δ fixed at $\delta = \text{const}$ and $\dot{\phi}$ fixed at zero. What was the classical Magnus term for the body of revolution becomes in effect a damping-in-roll term for the nonaxisymmetric body.

To summarize, with terms multiplied by $(q-\dot{\sigma})$ and $(r-\varepsilon\dot{\lambda})$ neglected, the aerodynamic moment system generalized to nonaxisymmetric bodies takes the form

$$\begin{aligned}C_k(t) = {} & C_k(\infty\,; \delta(t), \psi(t)) + (\dot{\psi}l/V)C_{k_{\dot{\psi}}}(\delta(t), \psi(t)) + \\ & (\dot{\sigma}l/V)C_{k_{\dot{\sigma}}}(\delta(t), \psi(t)) + (\dot{\lambda}l/V)(1/\gamma)C_{k_{\dot{\phi}}}(\infty\,; \delta(t), \psi(t)) \\ & k = l, m, n\end{aligned} \tag{14}$$

Unlike the body of revolution, further simplifications based on symmetry arguments are not possible except in special cases. Nevertheless, just as for the body of revolution, the moments due to an arbitrary motion may be compounded of the contributions from four simple motions: steady resultant angle of attack, oscillations in pitch at constant resultant angle of attack, coning at constant resultant angle of attack, and oscillations in roll at constant resultant angle of attack, all at a constant inclination of the body axes from the crossflow velocity vector.

Approximate formulation in the body axis system

The analysis based on the functional dependence specified by Eq. (9) parallels that of the previous sections. Expanded about $p_B = 0$, $q_B = 0$, $r_B = 0$, the integral form for $\hat{C}_m(t)$ yields, to first order in the rates

$$\begin{aligned}\hat{C}_m(t) = {} & \hat{C}_m(\infty\,; \hat{\alpha}(t), \hat{\beta}(t)) + (p_B l/V)\hat{C}_{m_{p_B}}(\infty\,; \hat{\alpha}(t), \hat{\beta}(t)) + \\ & (q_B l/V)\hat{C}_{m_{q_B}}(\infty\,; \hat{\alpha}(t), \hat{\beta}(t)) + (r_B l/V)\hat{C}_{m_{r_B}}(\infty\,; \hat{\alpha}(t), \hat{\beta}(t)) + \\ & (\dot{\hat{\alpha}}l/V)\hat{C}_{m_{\dot{\hat{\alpha}}}}(\hat{\alpha}(t), \hat{\beta}(t)) + (\dot{\hat{\beta}}l/V)\hat{C}_{m_{\dot{\hat{\beta}}}}(\hat{\alpha}(t), \hat{\beta}(t))\end{aligned} \tag{15}$$

where the zeros belonging to p_B, q_B, r_B have been omitted. Analogous expressions for $\hat{C}_l$ and $\hat{C}_n$ are obtained by substituting these coefficients for $\hat{C}_m$ in Eq. (15).

That Eqs. (11) and (15) yield compatible forms may be verified by transferring $\hat{C}_m$ and $\hat{C}_n$ to the aerodynamic axis system by the use of Eq. (10) and then replacing the variables $\dot{\hat{\alpha}}$, $\dot{\hat{\beta}}$, p_B, q_B, r_B by variables in the aerodynamic axis system through the use of Eqs. (3), (5), and (8). It will be found that each coefficient in $C_m(t)$ and $C_n(t)$ can be matched with a combination of coefficients in $\hat{C}_m(t)$ and $\hat{C}_n(t)$ having the same multiplying variable. The matches for the coefficients in $C_m(t)$ yield:

$$\left.\begin{aligned}C_m(\infty\,; \delta, \psi) &= \hat{C}_m(\infty\,; \hat{\alpha}, \hat{\beta})\cos\psi - \hat{C}_n(\infty\,; \hat{\alpha}, \hat{\beta})\sin\psi \\ C_{m_{\dot{\lambda}}}(\infty\,; \delta, \psi) &= \hat{C}_{m_{p_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\cos\psi - \hat{C}_{n_{p_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\sin\psi \\ C_{m_q}(\infty\,; \delta, \psi) &= \hat{C}_{m_{q_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\cos^2\psi + \hat{C}_{n_{r_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\sin^2\psi - \\ & \quad [\hat{C}_{n_{q_B}}(\infty\,; \hat{\alpha}, \hat{\beta}) + \hat{C}_{m_{r_B}}(\infty\,; \hat{\alpha}, \hat{\beta})]\cos\psi\sin\psi \\ C_{m_r}(\infty\,; \delta, \psi) &= \hat{C}_{m_{r_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\cos^2\psi - \hat{C}_{n_{q_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\sin^2\psi + \\ & \quad [\hat{C}_{m_{q_B}}(\infty\,; \hat{\alpha}, \hat{\beta}) - \hat{C}_{n_{r_B}}(\infty\,; \hat{\alpha}, \hat{\beta})]\cos\psi\sin\psi \\ C_{m_{\dot{\delta}}}(\delta, \psi) &= \hat{C}_{m_{\dot{\hat{\alpha}}}}(\hat{\alpha}, \hat{\beta})\cos^2\psi - \hat{C}_{n_{\dot{\hat{\beta}}}}(\hat{\alpha}, \hat{\beta})\sin^2\psi + \\ & \quad [\hat{C}_{m_{\dot{\hat{\beta}}}}(\hat{\alpha}, \hat{\beta}) - \hat{C}_{n_{\dot{\hat{\alpha}}}}(\hat{\alpha}, \hat{\beta})]\cos\psi\sin\psi \\ C_{m_{\dot{\psi}}}(\delta, \psi) &= \hat{C}_{m_{p_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\cos\psi - \hat{C}_{n_{p_B}}(\infty\,; \hat{\alpha}, \hat{\beta})\sin\psi + \\ & \quad \delta\hat{C}_{m_{\dot{\hat{\beta}}}}(\hat{\alpha}, \hat{\beta})\cos^2\psi + \delta\hat{C}_{n_{\dot{\hat{\alpha}}}}(\hat{\alpha}, \hat{\beta})\sin^2\psi - \\ & \quad \delta[\hat{C}_{m_{\dot{\hat{\alpha}}}}(\hat{\alpha}, \hat{\beta}) + \hat{C}_{n_{\dot{\hat{\beta}}}}(\hat{\alpha}, \hat{\beta})]\cos\psi\sin\psi\end{aligned}\right\} \tag{16}$$

The analogous matches for the coefficients in $C_n(t)$ may be obtained from Eq. (16) by replacing $\hat{C}_{m_i}$ with $\hat{C}_{n_i}$ and $\hat{C}_{n_i}$ with $-\hat{C}_{m_i}$. It is of particular interest to obtain the matches for the damping-in-pitch coefficient $C_{m_{\dot{\sigma}}}$, and the side-moment coefficients $C_{n_{\dot{\phi}}}$ and $(C_{n_{\dot{\phi}}} - \gamma C_{n_{\dot{\psi}}})$. Equations (13) and (16) yield

$$\begin{aligned}C_{m_{\dot{\sigma}}}(\delta, \psi) &= C_{m_q} + \gamma C_{m_{\dot{\delta}}} \\ &= (\hat{C}_{m_{q_B}} + \gamma\hat{C}_{m_{\dot{\hat{\alpha}}}})\cos^2\psi + (\hat{C}_{n_{r_B}} - \gamma\hat{C}_{n_{\dot{\hat{\beta}}}})\sin^2\psi - \\ & \quad - [(\hat{C}_{n_{q_B}} + \gamma\hat{C}_{n_{\dot{\hat{\alpha}}}}) + (\hat{C}_{m_{r_B}} - \gamma\hat{C}_{m_{\dot{\hat{\beta}}}})]\cos\psi\sin\psi\end{aligned} \tag{17}$$

$$\begin{aligned}C_{n_{\dot{\phi}}}(\infty\,; \delta, \psi) &= \gamma C_{n_{\dot{\lambda}}} + \delta C_{n_r} \\ &= \gamma(\hat{C}_{n_{p_B}}\cos\psi + \hat{C}_{m_{p_B}}\sin\psi) + \\ & \quad \delta[\hat{C}_{n_{r_B}}\cos^2\psi + \hat{C}_{m_{q_B}}\sin^2\psi + \\ & \quad (\hat{C}_{n_{q_B}} + \hat{C}_{m_{r_B}})\cos\psi\sin\psi]\end{aligned} \tag{18}$$

$$\begin{aligned}C_{n_{\dot{\phi}}} - \gamma C_{n_{\dot{\psi}}} = {} & \delta\{(\hat{C}_{m_{q_B}} + \gamma\hat{C}_{m_{\dot{\hat{\alpha}}}})\sin^2\psi + (\hat{C}_{n_{r_B}} - \gamma\hat{C}_{n_{\dot{\hat{\beta}}}})\cos^2\psi + \\ & [(\hat{C}_{n_{q_B}} + \gamma\hat{C}_{n_{\dot{\hat{\alpha}}}}) + (\hat{C}_{m_{r_B}} - \gamma\hat{C}_{m_{\dot{\hat{\beta}}}})]\cos\psi\sin\psi\}\end{aligned} \tag{19}$$

Combining Eqs. (17) and (19) yields the following interesting equality:

$$C_{n_{\dot{\phi}}} - \gamma C_{n_{\dot{\psi}}} + \delta C_{m_{\dot{\sigma}}} = \delta[(\hat{C}_{m_{q_B}} + \gamma\hat{C}_{m_{\dot{\hat{\alpha}}}}) + (\hat{C}_{n_{r_B}} - \gamma\hat{C}_{n_{\dot{\hat{\beta}}}})] \tag{20}$$

The term $(\hat{C}_{m_{q_B}} + \gamma\hat{C}_{m_{\dot{\hat{\alpha}}}})$ is recognized as being the planar damping-in-pitch coefficient measured along y_B for small oscillations in $\hat{\alpha}$ about $\hat{\alpha} = \text{const}$, with $\hat{\beta}$ held fixed at $\hat{\beta} = \text{const}$. Similarly, $(\hat{C}_{n_{r_B}} - \gamma\hat{C}_{n_{\dot{\hat{\beta}}}})$ is the damping-in-yaw coefficient measured

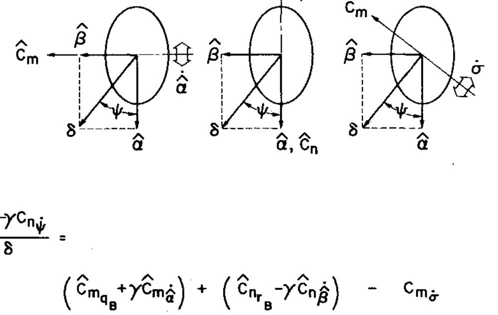

Fig. 2 Schematic representation of the equality between $(C_{n_{\dot{\phi}}} - \gamma C_{n_{\dot{\psi}}})/\delta$ and the three damping coefficients.

along z_B for small oscillations in $\hat{\beta}$ about $\hat{\beta}$ = const, with $\hat{\alpha}$ held fixed at $\hat{\alpha}$ = const. Thus, a measurement for $(C_{n_{\dot{\phi}}}-\gamma C_{n_{\dot{\psi}}})$ would be equivalent to a measure of a combination of the three damping coefficients. The identity is shown schematically in Fig. 2. Equation (20) generalizes to the nonlinear case and to arbitrary bodies the relationship between $(C_{n_{\dot{\phi}}}-\gamma C_{n_{\dot{\psi}}})$ and the damping coefficients that was pointed out and verified for bodies of revolution in the linear case in Ref. 5. It is noted that $(C_{n_{\dot{\phi}}}-\gamma C_{n_{\dot{\psi}}})$ equals $\delta(\hat{C}_{m_{q_B}}+\gamma\hat{C}_{m_{\dot{\hat{\alpha}}}})$ when $\hat{\alpha}=0$, and equals $\delta(\hat{C}_{n_{r_B}}-\gamma\hat{C}_{n_{\dot{\hat{\beta}}}})$ when $\hat{\beta}=0$. Under conditions where a linear formulation of the moment system can be assumed to hold (e.g., when $\hat{\alpha}\to 0$, $\hat{\beta}\to 0$), it is consistent to assume that the couplings between motions in $\hat{\alpha}$ and $\hat{\beta}$ will be negligibly small. Under these conditions, the measurement of $(C_{n_{\dot{\phi}}}-\gamma C_{n_{\dot{\psi}}})$ at $\hat{\alpha}=0$ and again at $\hat{\beta}=0$ is all that is required to yield measures of the damping coefficients characteristic of the two uncoupled modes.

Simplification of the formulation in the body axis system

Equation (15) can be simplified for the case of small plunging to yield a form analogous to that of Eq. (14) in the aerodynamic axis system. The approximate expressions for q_B and r_B, consistent with $q \approx \dot{\sigma}$, $r \approx \varepsilon\dot{\lambda}$, are

$$\left.\begin{aligned} q_B &\approx (1/\gamma)\dot{\hat{\alpha}}+(\hat{\beta}/\gamma)p_B \\ r_B &\approx -(1/\gamma)\dot{\hat{\beta}}+(\hat{\alpha}/\gamma)p_B \end{aligned}\right\} \tag{21}$$

Substituting in Eq. (15) gives

$$\begin{aligned}\hat{C}_m(t) = {} & \hat{C}_m(\infty;\hat{\alpha},\hat{\beta})+(1/\gamma)(p_B l/V)[\gamma\hat{C}_{m_{p_B}}(\infty;\hat{\alpha},\hat{\beta})+ \\ & \hat{\beta}\hat{C}_{m_{q_B}}(\infty;\hat{\alpha},\hat{\beta})+\hat{\alpha}\hat{C}_{m_{r_B}}(\infty;\hat{\alpha},\hat{\beta})]+(1/\gamma)(\dot{\hat{\alpha}}l/V)\times \\ & [\hat{C}_{m_{q_B}}(\infty;\hat{\alpha},\hat{\beta})+\gamma\hat{C}_{m_{\dot{\hat{\alpha}}}}(\hat{\alpha},\hat{\beta})]-(1/\gamma)(\dot{\hat{\beta}}l/V)\times \\ & [\hat{C}_{m_{r_B}}(\infty;\hat{\alpha},\hat{\beta})-\gamma\hat{C}_{m_{\dot{\hat{\beta}}}}(\hat{\alpha},\hat{\beta})]\end{aligned} \tag{22}$$

The first term is the pitching-moment coefficient along y_B that would be measured in steady planar motion with $\hat{\alpha}$ and $\hat{\beta}$ at the fixed inclinations $\hat{\alpha}$ = const, $\hat{\beta}$ = const. The combination of terms multiplied by $p_B l/V\gamma$ can be shown [from Eq. (18)] to be the rate of change with $\dot{\phi}l/V$, evaluated at $\dot{\phi}=0$, of the pitching-moment coefficient along y_B that would be measured in a steady coning motion $\hat{\alpha}$ = const, $\hat{\beta}$ = const, $\dot{\phi}$ = const. The term $(\hat{C}_{m_{r_B}}-\gamma\hat{C}_{m_{\dot{\hat{\beta}}}})$ is a cross-coupling term resulting from motion in the $\hat{\beta}$ plane. This term and the analogous term in $\hat{C}_n(t)$, $(\hat{C}_{n_{q_B}}+\gamma\hat{C}_{n_{\dot{\hat{\alpha}}}})$, are the nonlinear interaction terms that are normally excluded in the classical treatment and are missed by attempts to generalize from linear formulations based on the principle of superposition. The aerodynamic moment system in body axes takes the form

$$\begin{aligned}\hat{C}_k(t) = {} & \hat{C}_k(\infty;\hat{\alpha},\hat{\beta})+(1/\gamma)(p_B l/V)\hat{C}_{k_{\dot{\phi}}}(\infty;\hat{\alpha},\hat{\beta})+ \\ & (1/\gamma)(\dot{\hat{\alpha}}l/V)[\hat{C}_{k_{q_B}}(\infty;\hat{\alpha},\hat{\beta})+\gamma\hat{C}_{k_{\dot{\hat{\alpha}}}}(\hat{\alpha},\hat{\beta})]- \\ & (1/\gamma)(\dot{\hat{\beta}}l/V)[\hat{C}_{k_{r_B}}(\infty;\hat{\alpha},\hat{\beta})-\gamma\hat{C}_{k_{\dot{\hat{\beta}}}}(\hat{\alpha},\hat{\beta})] \\ & \qquad k = l, m, n\end{aligned} \tag{23}$$

In the body axis system, the four characteristic motions are: steady angle of attack and sideslip, coning at constant angle of attack and sideslip, and the oscillations in pitch and yaw at constant angles of attack and sideslip. The oscillations-in-roll experiment that was required in the aerodynamic axis system is, in effect, incorporated in the oscillations in pitch and yaw experiments in the body axis system. It will be noted that, for example, oscillating in $\hat{\alpha}$ with $\hat{\beta}$ fixed involves an oscillation in ψ.

Discussion

Within the assumption of a linear dependence of the moment on coning rate, the analysis presented here suggests that the moment contributions resulting from four characteristic motions are required to completely specify the nonlinear moment system for arbitrary motions. For wind-tunnel tests in the aerodynamic axis system, two kinds of apparatus would appear to be necessary: 1) A coning and spinning apparatus similar to the one described in Ref. 5. (For nonaxisymmetric bodies, the spin motor that reproduced the constant spin rate $\dot{\psi}$ would have to be replaced by a device reproducing small oscillations in ψ about ψ = const.) Such an apparatus should be capable of measuring the moment contributions due to steady resultant angle of attack, coning at constant resultant angle of attack, and oscillations in roll at constant resultant angle of attack; 2) An oscillations-in-pitch apparatus for measuring the moments due to small oscillations in σ about a fixed σ with the axis of rotation oriented normal to the σ plane. It is emphasized that for nonaxisymmetric bodies, the oscillations-in-pitch apparatus must be capable of measuring not only the pitching moment but also the induced side and rolling moments.

Wind-tunnel tests in the body axis system require the same coning experiment as in the aerodynamic axis system and, in addition, separate oscillations-in-pitch and oscillations-in-yaw experiments. It may be worth repeating that a separate oscillations-in-roll experiment is not required in the body axis system, the roll motion being incorporated in the oscillations-in-pitch and oscillations-in-yaw experiments. The oscillations-in-pitch device must be capable of simultaneously measuring all three moment components due to small oscillations in $\hat{\alpha}$ about a fixed $\hat{\alpha}$ with $\hat{\beta}$ held fixed. The same device can be used in the oscillations-in-yaw experiment, in which the roles of $\hat{\alpha}$ and $\hat{\beta}$ are reversed. Hence, experiments carried out in the body axis system, requiring only a single oscillatory device, would appear to have an advantage over those in the aerodynamic axis system from the standpoint of economics. It is essential, however, that the oscillatory experiments in the body axis system enable the measurement of the nonlinear interaction terms *even for bodies of revolution.* Recently, the successful development of a device capable of measuring these interaction terms has been reported.[6]

In programs designed to extract the nonlinear aerodynamic coefficients from free-flight data, it is recommended of course that a form for the aerodynamic moment system based on Eq. (14) or Eq. (23) be incorporated. It has already been noted[2] that procedures not allowing for the presence of interaction terms in the representation of the moment system can assign erroneous weights to the remaining terms.

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